MATCHED FILTERS

- The matched filter is the optimal linear filter for maximizing the signal to noise ratio (SNR) in the presence of additive stochastic noise.
- Matched filters are commonly used in radar, in which a signal is sent out, and we measure the reflected signals, looking for something similar to what was sent out.
- Two-dimensional matched filters are commonly used in image processing, e.g., to improve SNR for X-ray pictures

- A general representation for a matched filter is illustrated in Figure 3-1

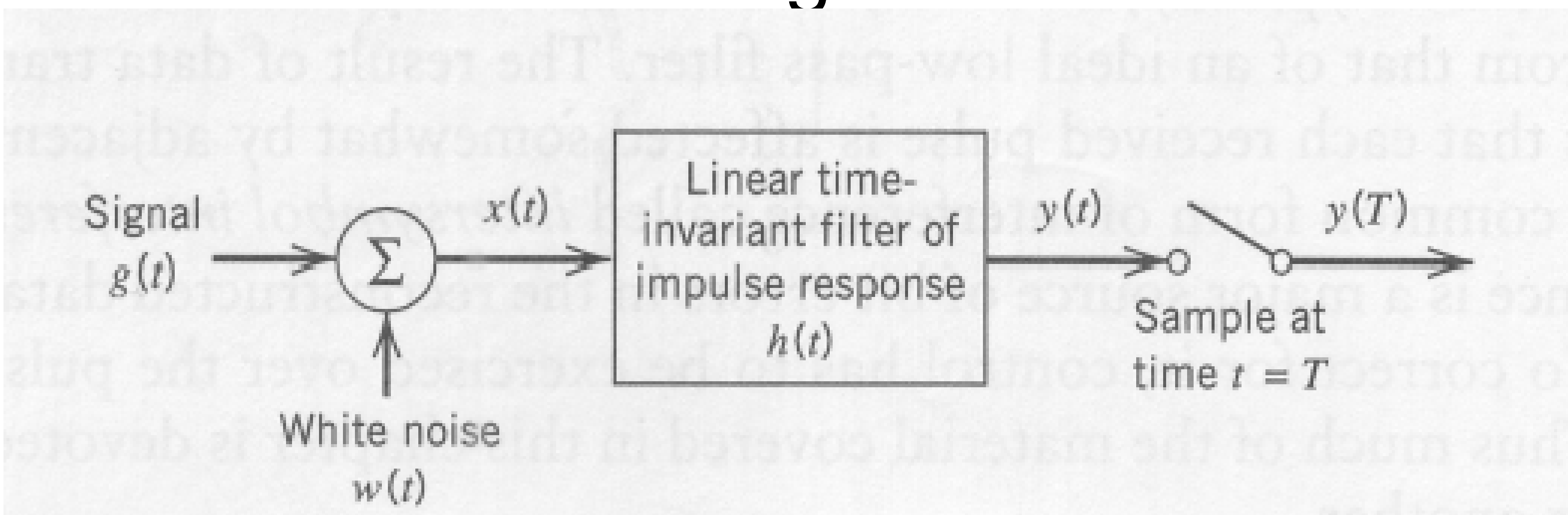

FIGURE 3-1 Matched filter

- The filter input *x(t)* consists of a pulse signal *g(t)* corrupted by additive channel noise w(t), as shown by

$$x(t) = g(t) + w(t), \qquad 0 \leq t \leq T \qquad \textbf{(3.0)}$$

- where T is an arbitrary observation interval. The pulse signal *g(t)* may represent a binary symbol I or 0 in a digital communication system.
- The *w(t) is* the sample function of a white noise process of zero mean and power spectral density No/2.
- The source of uncertainty lies in the noise w(t).
- The function of the receiver is to detect the pulse signal *g(t)* in an optimum manner, given the received signal x(t).
- To satisfy this requirement, we have to optimize the design of the filter so as to minimize the effects of noise at the filter output in some statistical sense, and thereby enhance the detection of the pulse signal g(t).

- Since the filter is linear, the resulting output *y(t)* may be expressed as
- $$y(t) = g_o(t) + n(t) \qquad (3.1)$$

- where $g_o(t)$ and *n(t)* are produced by the signal and noise components of the input *x(t),* respectively.
- A simple way of describing the requirement that the output signal component $g_o(t)$ be considerably greater than the output noise component *n(t) is* to have the filter make the instantaneous power in the output signal $g_o(t)$, measured at time $t = T$, as large as possible compared with the average power of the output noise n(t). This is equivalent to maximizing the *peak pulse signal-to-noise ratio,* defined as

$$\eta = \frac{|g_o(T)|^2}{E[n^2(t)]} \qquad (3.2)$$

White Noise

- For the case of white noise, the description of the matched filter is simplified as follows: For white noise, = No / 2. Thus equation becomes,

$$H(f) = \frac{2K}{N_0} S^*(f) e^{-j\omega t_0} \qquad \textbf{(3.3)}$$

- From this equation, the following theorem is obtained

Example 3.1

THEOREM. *When the input noise is white, the impulse response of the matched filter becomes*

$$h(t) = Cs(t_0 - t) \qquad \textbf{(3.4)}$$

where C is an arbitrary real positive constant, t_0 is the time of the peak signal output, and $s(t)$ is the known input-signal waveshape.

Proof

$$h(t) = \mathscr{F}^{-1}[H(f)] = \frac{2K}{N_0} \int_{-\infty}^{\infty} S^*(f)\; e^{-j\omega t_0} e^{j\omega t}\; df$$

$$= \frac{2K}{N_0} \left[\int_{-\infty}^{\infty} S(f) e^{j2\pi f(t_0 - t)}\; df \right]^*$$

$$= \frac{2K}{N_0} [s(t_0 - t)]^*$$

- But s(t) is a real signal; hence, let C=2K/No so that the impulse response is equivalent to equation (3.4).
- Equation (3.4) shows that the impulse response of the matched filter (white-noise case) is simply the known signal waveshape that is "played backward" and translated by an amount to. Thus, the filter is said to be "matched" to the signal.

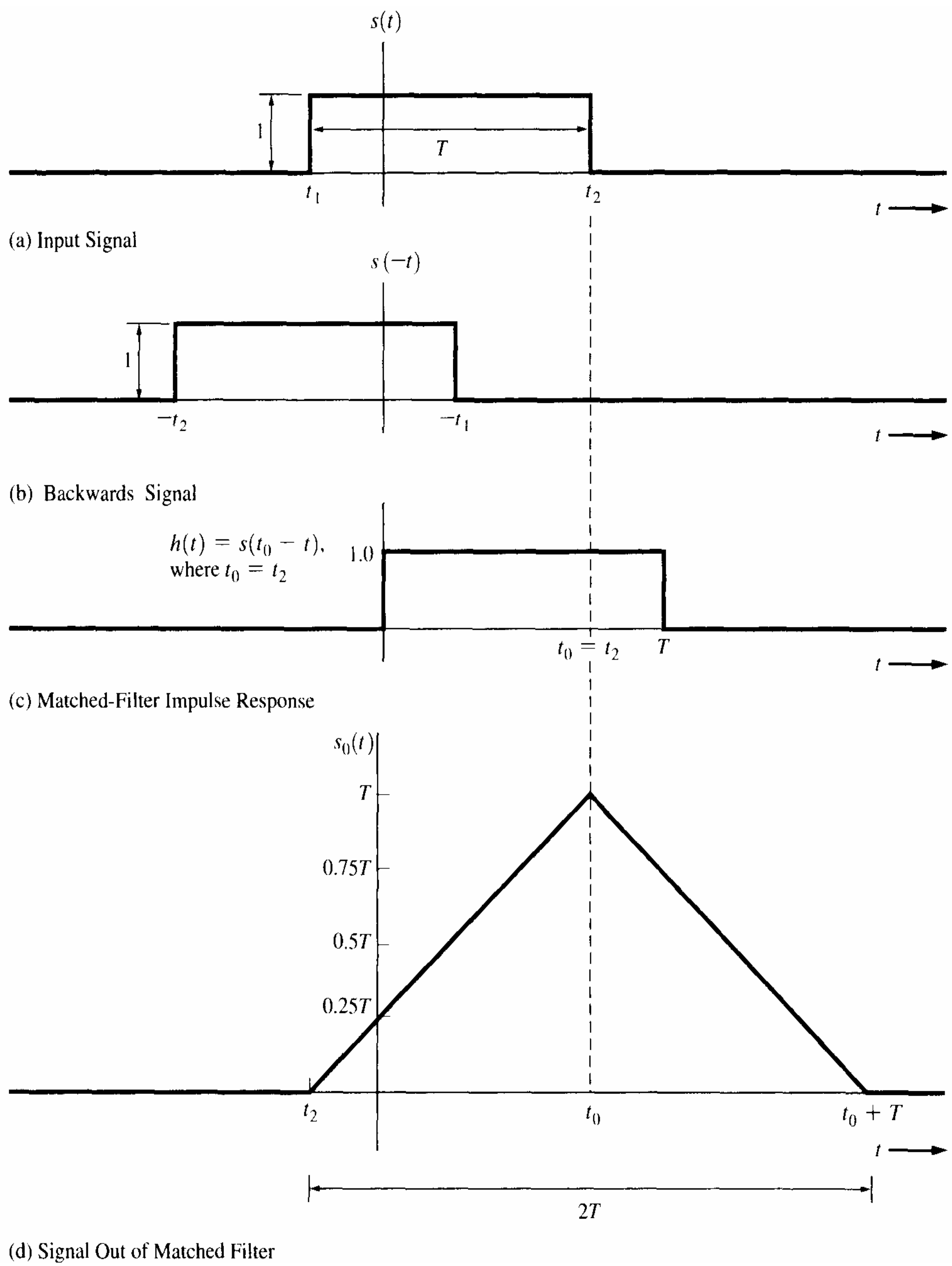

FIGURE 3-2 Waveforms associated with the match filter of Example 3-1

Error rate due to Noise

- To proceed with the analysis, consider a binary PCM system based on *polar nonreturn-to-zero (NRZ) signaling.*

- symbols 1 and 0 are represented by positive and negative rectangular pulses of equal amplitude and equal duration.

- The channel noise is modeled as *additive white Gaussian noise w(t)* of zero mean and power spectral density $N_O/2$; the Gaussian assumption is needed for later calculations. In the signaling interval $O < t < T_b$ the received signal written as:

$$x(t) = \begin{cases} +A + w(t), & \text{symbol 1 was sent} \\ -A + w(t), & \text{symbol 0 was sent} \end{cases}$$

- where *Tb* is the *bit duration,* and A is the *transmitted pulse amplitude.*

- The receiver has acquired knowledge of the starting and ending times of each transmitted pulse;

- Given the noisy signal x(t), the receiver is required to make a decision in each signaling interval as to whether the transmitted symbol is a 1 or a 0.

- The structure of the receiver used to perform this decision-making process is shown in Figure 3.3. It consists of a matched filter followed by a sampler, and then finally a decision device.

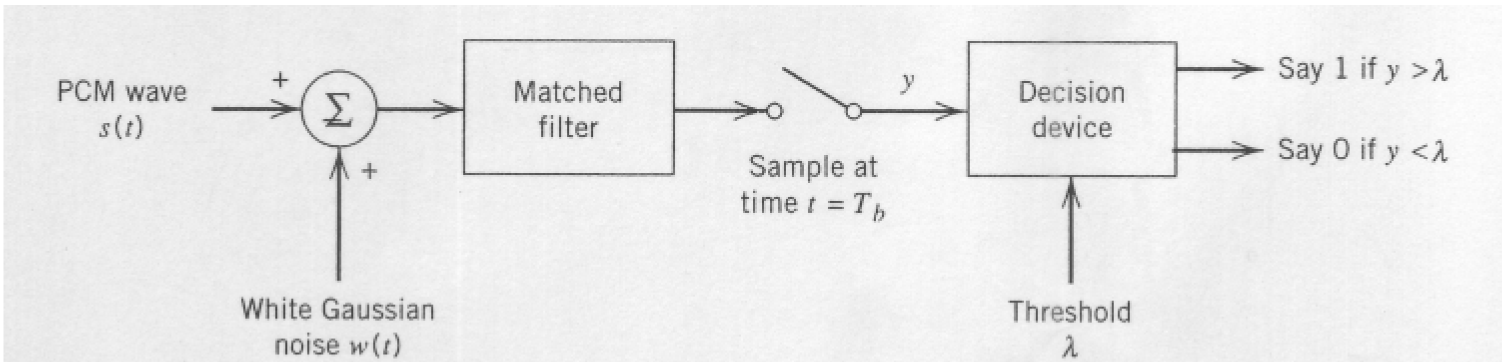

FIGURE 3.3 Receiver for baseband transmission of binary-encoded PCM wave using polar NRZ signaling.

The filter is matched to a rectangular pulse of amplitude A and duration T*b*, exploiting the bit-timing information available to the receiver.
The resulting matched filter output is sampled at the end of each signaling interval.

The presence of channel noise *w(t)* adds randomness to the matched filter output.

- The sample value y is compared to a preset *threshold* A in the decision device.

- If the threshold is exceeded, the receiver makes a decision in favor of symbol 1; if not, a decision is made in favor of symbol 0.

- There are two possible kinds of error to be considered:

  – 1. Symbol 1 is chosen when a 0 was actually transmitted; we refer to this error as an *error of the first kind.*

  – 2. Symbol 0 is chosen when a 1 was actually transmitted; we refer to this error as an *error o f the second kind.*

Raised Cosine Spectrum

- Overcome the practical difficulties encountered with the ideal Nyquist channel by extending the bandwidth from the minimum value W = $R_b/2$ to an adjustable value between W and 2W.

- Restrict the frequency band of interest to [- W, W], as shown by

$$P(f) + P(f - 2W) + P(f + 2W) = \frac{1}{2W}, \qquad -W \leq f \leq W$$

- A particular form of *P(f)* that embodies many desirable features is provided by a *raised cosine spectrum.*
- This frequency response consists of a *flat* portion and *a rollo f f* portion that has a sinusoidal form, as follows:

$$P(f) = \begin{cases} \dfrac{1}{2W}, & 0 \le |f| < f_1 \\ \dfrac{1}{4W}\left\{1 - \sin\left[\dfrac{\pi(|f| - W)}{2W - 2f_1}\right]\right\}, & f_1 \le |f| < 2W - f_1 \\ 0, & |f| \ge 2W - f_1 \end{cases}$$

- The frequency parameter f, and bandwidth W are related by

$$\alpha = 1 - \frac{f_1}{W}$$

- The parameter *a* is called the *rolloff factor;* it indicates the *excess bandwidth* over the ideal solution, W. Specifically, the transmission bandwidth BT is defined by

$$\begin{aligned} B_T &= 2W - f_1 \\ &= W(1 + \alpha) \end{aligned}$$

- The frequency response P(f), normalized by multiplying it by 2W, is plotted in Figure *3.4a* for three values of *a,* namely, 0, 0.5, and 1.

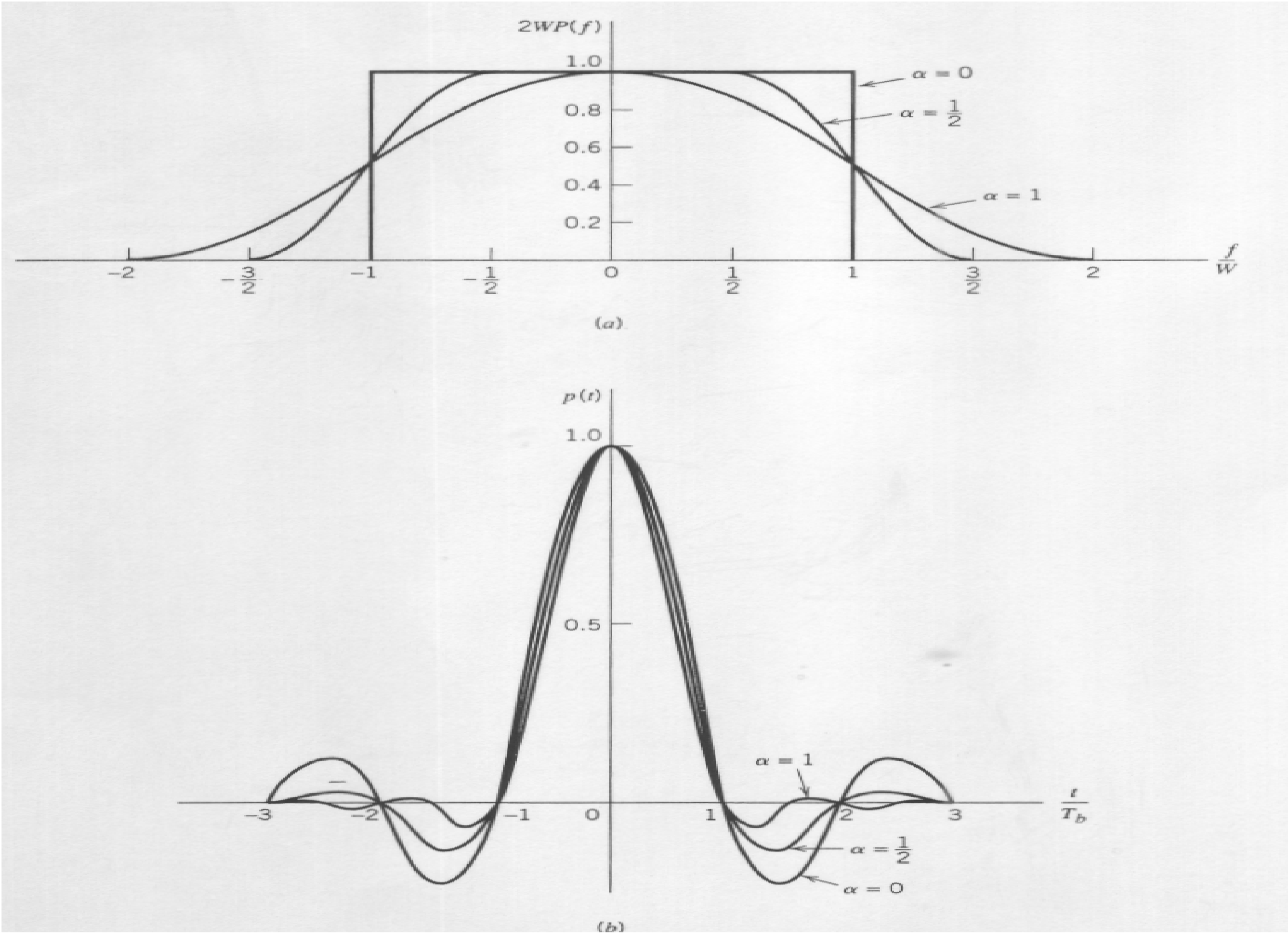

FIGURE 3-4 Responses for different roll-off factors.
(a) Frequency response. (b) Time response.

TABLE A6.6 *The error function*[a]

u	$erf(u)$	u	$erf(u)$
0.00	0.00000	1.10	0.88021
0.05	0.05637	1.15	0.89612
0.10	0.11246	1.20	0.91031
0.15	0.16800	1.25	0.92290
0.20	0.22270	1.30	0.93401
0.25	0.27633	1.35	0.94376
0.30	0.32863	1.40	0.95229
0.35	0.37938	1.45	0.95970
0.40	0.42839	1.50	0.96611
0.45	0.47548	1.55	0.97162
0.50	0.52050	1.60	0.97635
0.55	0.56332	1.65	0.98038
0.60	0.60386	1.70	0.98379
0.65	0.64203	1.75	0.98667
0.70	0.67780	1.80	0.98909
0.75	0.71116	1.85	0.99111
0.80	0.74210	1.90	0.99279
0.85	0.77067	1.95	0.99418
0.90	0.79691	2.00	0.99532
0.95	0.82089	2.50	0.99959
1.00	0.84270	3.00	0.99998
1.05	0.86244	3.30	0.999998

[a]The error function is tabulated extensively in several references; see for example, Abramowitz and Stegun (1965, pp. 297–316).